Reducibility proofs in $\lambda$-calculi with intersection types

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By using reducibility, new, simple and general methods can be developed to prove properties of the $\lambda$-calculus.

In our paper:
- We review and find the flaws in one reducibility method of proofs of Church-Rosser, standardisation and weak head normalisation.
- We review, adapt and non trivially extend another reducibility method of proofs of Church-Rosser.
The Two Reducibility Methods

1. Ghilezan and Likavec’s method:
   ➤ According to this method, a certain property of the $\lambda$-calculus is proved to hold, if that property satisfies a certain set of predicates.
   ➤ Unfortunately, this method does not work. We give counterexamples.

2. Koletsos and Stavrinos’s method:
   ➤ This method aims to prove the Church-Rosser property of the untyped $\lambda$-calculus by showing first that a typed $\lambda$-calculus is confluent and using this to show the confluence of developments.
   ➤ We adapt this method to $\beta I$-reduction.
   ➤ We extend (this is non trivial) this method to $\beta \eta$-reduction.
Ghilezan and Likavec’s Method [GL02]

Ghilezan and Likavec designed a general proof method schema.

The basic step of the method: if a set of λ-terms $\mathcal{P}$ satisfies a defined set of predicates $\text{pred}$ then it contains a certain set of typable λ-terms $T$.

\[ \text{pred}(\mathcal{P}) \Rightarrow T \subseteq \mathcal{P} \]

Extension of the basic step: if a set of λ-terms $\mathcal{P}$ satisfies a defined set of predicates $\text{pred}$ then it contains the whole set of λ-terms.

\[ \text{pred}(\mathcal{P}) \Rightarrow \Lambda = \mathcal{P} \]
Below, $\mathcal{P}$ is a set of terms. Using:

- a set of types $\sigma \in \text{Type}^1 := \alpha | \sigma_1 \rightarrow \sigma_2 | \sigma_1 \cap \sigma_2$,
- a type interpretation function $\lbrack - \rbrack^1_\mathcal{P}$ which depends on $\mathcal{P}$ and
- a set of predicates $\text{pred}$ which depends on type interpretations and consists of:
  - Variable predicate: each variable belongs to each type interpretation.
  - Saturation predicate (1): the contractum of a $\beta$-redex is in a type interpretation $\Rightarrow$ the $\beta$-redex is in the type interpretation.
  - Closure predicate (1): a term applied to a variable is in a type interpretation $\Rightarrow$ the term is in the set of terms given as parameter.

Ghilezan and Likavec claim that $\text{pred}(\mathcal{P}) \Rightarrow \text{SN} \subseteq \mathcal{P}$.

(where $\text{SN} = \{ M | \text{each reduction from } M \text{ is finite} \} = \text{set of } \lambda\text{-terms typable in } D$).
Recall that $\mathcal{P}$ is a set of terms. Using:

- a set of types $\tau \in \text{Type}^2 ::= \alpha \mid \tau_1 \rightarrow \tau_2 \mid \tau_1 \cap \tau_2 \mid \Omega$,
- a type interpretation depending on $\mathcal{P}$,
- a set of predicates $\text{pred}$ which depends on type interpretations and consists of:
  - Variable predicate: same as before.
  - Saturation predicate (2): similar to before.
  - Closure predicate (2): a term is in a type interpretation $\Rightarrow$ the abstraction of the term is in $\mathcal{P}$.
- an intersection type system (with omega and subtyping rule),

Ghilezan and Likavec prove that $\text{pred}(\mathcal{P}) \Rightarrow T \subseteq \mathcal{P}$

where $T$ is a set of typable terms under some restriction on types.
Ghilezan and Likavec’s method [GL02]
full method- basic step continued

▶ It is not easy to prove \( \text{pred}(P) \). Hence, [GL02] introduces:
  ➢ stronger induction hypotheses. These are new predicates collected in a set newpred.
  ➢ These new predicates do not deal with type interpretation

▶ newpred(CR) where  
\[
\text{CR} = \{M \mid M \rightarrow^* M_1 \land M \rightarrow^* M_2 \Rightarrow \exists M'. M_1 \rightarrow^*_\beta M' \land M_2 \rightarrow^*_\beta M' \}
\]

▶ newpred(W) where  
\[
\text{W} = \{M \mid \exists n \in \mathbb{N}. \exists x \in V. \exists M, M_1, \ldots, M_n \in \Lambda. (M \rightarrow^*_\beta \lambda x.M \lor M \rightarrow^*_\beta xM_1 \ldots M_n) \}
\]

▶ newpred(S) where  
\[
\text{S} = \{M \mid M \rightarrow^*_\beta M' \Rightarrow \exists N. M \rightarrow^*_h N \land N \rightarrow^*_i M' \} (\rightarrow^*_h \text{ for head-reduction and } \rightarrow^*_i \text{ for internal-reduction})
\]
The final step of the method is to prove
\[ \text{newpred}(\mathcal{P}) \land \text{Inv}(\mathcal{P}) \Rightarrow \Lambda = \mathcal{P} \]
where \( \Lambda \) is the set of all the \( \lambda \)-terms and
\textbf{Invariance predicate Inv:}
If \( M \in \Lambda \) then \( \lambda x. M \in \mathcal{P} \iff M \in \mathcal{P} \).

The authors give a set \( T \) of \( \lambda \)-terms that are typable in their type system with a type satisfying the necessary restrictions.

This final step is done in two parts:

- Let \( M \in \Lambda \). Then:
  - \( \lambda x. M \in T \)
  - \( \text{newpred}(\mathcal{P}) \Rightarrow \lambda x. M \in \mathcal{P} \)
  - \( \text{newpred}(\mathcal{P}) \land \text{Inv}(\mathcal{P}) \Rightarrow M \in \mathcal{P} \)
- \( \text{Inv(CR)} \) and \( \text{Inv(S)} \).
Ghilezan and Likavec’s method fails

Counterexample

- Our paper lists in detail the problems with a number of lemmas and proofs in [GL02].
- Here, we show one counterexample:

Claim [GL02]

\[
\text{INV}(\mathcal{P}) \land \text{VAR}(\mathcal{P}) \land \text{SAT}(\mathcal{P}) \Rightarrow \Lambda = \mathcal{P}.
\]

Counter-example: INV(WN), VAR(WN) and SAT(WN) are true, but WN \neq \Lambda.
Ghilezan and Likavec’s method [GL02]

summary

First step:

\[ \text{pred1}(\mathcal{P}) \Rightarrow T \subseteq \mathcal{P}. \]

(where \( T \) is a set of typable terms in a given type system)

Full method (false):

\[ \text{pred2}(\mathcal{P}) \Rightarrow \Lambda = \mathcal{P}. \]

We tried to salvage the full method of Ghilezan and Likavec, but we failed. We did not go further than the basic step with \( T = \text{SN} \), which is a result Ghilezan and Likavec already proved.

Some similar proof methods have already been, as far as we know, successfully developed (for example by Gallier [Gal03]). However, they do not go further than the basic step and do not deal with Church-Rosser. Such methods can help in characterising typable terms w.r.t. a type system.
Koletsos and Stavrinos’s method [KS08]
the outlines of their method
Koletsos and Stavrinos’s method [KS08] proves Church Rosser of $\beta$-reduction.

We extend Koletsos and Stavrinos’s method to prove Church Rosser of $\beta\eta$-reduction.

$$\text{CRBE} = \{ M \mid M \rightarrow^*_{\beta\eta} M_1 \land M \rightarrow^*_{\beta\eta} M_2 \Rightarrow \exists M'. M_1 \rightarrow^*_{\beta\eta} M' \land M_2 \rightarrow^*_{\beta\eta} M' \}$$

Using:
- a set of types,
- a type system,
- a type interpretation based on CRBE and
- a language typable in the type system,

we prove that each term in the defined language is in CRBE.
What is this new language? The parametrised language $\Lambda \eta_c \subseteq \Lambda$ is defined as follows:

1. If $x$ is a variable distinct from $c$ then
   - $x \in \Lambda \eta_c$.
   - If $M \in \Lambda \eta_c$ then $\lambda x. (M[x := c(cx)]) \in \Lambda \eta_c$.
   - If $N x \in \Lambda \eta_c$, $x \notin \text{fv}(N)$ and $N \neq c$ then $\lambda x. N x \in \Lambda \eta_c$.

2. If $M, N \in \Lambda \eta_c$ then $cMN \in \Lambda \eta_c$.

3. If $M, N \in \Lambda \eta_c$ and $M$ is a $\lambda$-abstraction then $MN \in \Lambda \eta_c$.

4. If $M \in \Lambda \eta_c$ then $cM \in \Lambda \eta_c$. 
An Extension of Koletsos and Stavrinos’s method [KS08]
a bit a technicality

\[ p \in \text{Path} ::= 0 \mid 1.p \mid 2.p. \]

We define \( M|_p \) as follows:

- \( M|_0 = M \)
- \( (\lambda x.M)|_{1.p} = M|_p \)
- \( (MN)|_{1.p} = M|_p \)
- \( (MN)|_{2.p} = N|_p \).

Example: \( (\lambda x.zx)|_{1.2.0} = (zx)|_{2.0} = x|_0 = x. \)
An Extension of Koletsos and Stavrinos’s method [KS08]  
a bit a technicality

Let us define the three following common relations:

- $\beta ::= \langle (\lambda x. M) N, M[x := N] \rangle$
- $\eta ::= \langle \lambda x. M x, M \rangle$, where $x \not\in \text{FV}(M)$
- $\beta\eta = \beta \cup \eta$

Let $r \in \{\beta, \eta, \beta\eta\}$

$R^r = \{L \mid \langle L, R \rangle \in r\}$ and $R^r_M = \{p \mid M|_p \in R^r\}$

Example: $R^\beta_{(\lambda x. y x)y} = \{0, 1.0\}$.

We define the ternary relation $\rightarrow_r$ as follows:

- $M \xrightarrow{0}_r M'$ if $\langle M, M' \rangle \in r$
- $\lambda x. M \xrightarrow{1,.p}_r \lambda x. M'$ if $M \xrightarrow{p}_r M'$
- $MN \xrightarrow{1,.p}_r M'N$ if $M \xrightarrow{p}_r M'$
- $NM \xrightarrow{2,.p}_r NM'$ if $M \xrightarrow{p}_r M'$
- $M \xrightarrow{p}_r M'$ if there exists $p$ such that $M \xrightarrow{p}_r M'$.

Example: $(\lambda x. x)y \xrightarrow{0}_\beta y \Rightarrow \lambda y.(\lambda x. x)y \xrightarrow{1.0}_\beta \lambda y. y$. 

An Extension of Koletsos and Stavrinos’s method [KS08]

a bit a technicality - An erasure function

Erasure on terms:

▶ \(|x|^c = x\)
▶ \(|\lambda x.N|^c = \lambda x.|N|^c, \text{ if } x \neq c\)
▶ \(|cP|^c = |P|^c\)
▶ \(|NP|^c = |N|^c|P|^c, \text{ if } N \neq c\)

Example: \(|(c(\lambda x.yx))y|^c = (\lambda x.yx)y\).

Erasure on paths:

▶ \(|\langle M, 0 \rangle|^c = 0\)
▶ \(|\langle \lambda x.M, 1.p \rangle|^c = 1.|\langle M, p \rangle|^c, \text{ if } x \neq c\)
▶ \(|\langle MN, 1.p \rangle|^c = 1.|\langle M, p \rangle|^c\)
▶ \(|\langle cM, 2.p \rangle|^c = |\langle M, p \rangle|^c\)
▶ \(|\langle NM, 2.p \rangle|^c = 2.|\langle M, p \rangle|^c, \text{ if } N \neq c\)

Example: \(|\langle (c(\lambda x.yx))y, 1.2.0 \rangle|^c = 1.0\).
An Extension of Koletsos and Stavrinos’s method [KS08]
a bit a technicality - a function from $\Lambda \times 2^{\text{Path}}$ to $2^{\Lambda \eta}$

Let $c \not\in \text{fv}(M)$ and $F \subseteq R^\beta_\eta_M$.

1. If $M \in V \setminus \{c\}$ then $F = \emptyset$ and
   \[
   \Psi^c(M, F) = \{c^n(M) \mid n > 0\}
   \]
   \[
   \Psi^c_0(M, F) = \{M\}
   \]

2. If $M = \lambda x.N$ and $x \neq c$ and $F' = \{p \mid 1.p \in F\} \subseteq R^\beta_\eta_N$ then:
   \[
   \Psi^c(M, F) =
   \begin{cases}
   \{c^n(\lambda x.P[x := c(cx)]) \mid n \geq 0 \land P \in \Psi^c(N, F')\} & \text{if } 0 \not\in F \\
   \{c^n(\lambda x.N') \mid n \geq 0 \land N' \in \Psi^c_0(N, F')\} & \text{otherwise}
   \end{cases}
   \]
   \[
   \Psi^c_0(M, F) =
   \begin{cases}
   \{\lambda x.N'[x := c(cx)] \mid N' \in \Psi^c(N, F')\} & \text{if } 0 \not\in F \\
   \{\lambda x.N' \mid N' \in \Psi^c_0(N, F')\} & \text{otherwise}
   \end{cases}
   \]

3. If $M = NP$, $F_1 = \{p \mid 1.p \in F\} \subseteq R^\beta_\eta_N$ and $F_2 = \{p \mid 2.p \in F\} \subseteq R^\beta_\eta_P$ then:
   \[
   \Psi^c(M, F) =
   \begin{cases}
   \{c^n(cN'P') \mid n \geq 0 \land N' \in \Psi^c(N, F_1) \land P' \in \Psi^c(P, F_2)\} & \text{if } 0 \not\in F \\
   \{c^n(N'P') \mid n \geq 0 \land N' \in \Psi^c_0(N, F_1) \land P' \in \Psi^c(P, F_2)\} & \text{otherwise}
   \end{cases}
   \]
   \[
   \Psi^c_0(M, F) =
   \begin{cases}
   \{cN'P' \mid N' \in \Psi^c(N, F_1) \land P' \in \Psi^c_0(P, F_2)\} & \text{if } 0 \not\in F \\
   \{N'P' \mid N' \in \Psi^c_0(N, F_1) \land P' \in \Psi^c_0(P, F_2)\} & \text{otherwise}
   \end{cases}
   \]
Example:

\[
\Psi^c((\lambda x.(\lambda y.M)x)N, \{1, 1.0, 1.1.0\}) = \\
\{c^n((\lambda x.(\lambda y.P[y := c(cy)])x)Q) \mid n \geq 0 \land P \in \Psi^c(M, \emptyset) \land Q \in \Psi^c(N, \emptyset)\} \subseteq \Lambda \eta_c,
\]

where \(x \notin \text{fv}(\lambda y.M)\).

Let \(p = 1.0\) then \((\lambda x.(\lambda y.M)x)N \overrightarrow{\beta \eta} (\lambda y.M)N\).

Let \(n \geq 0, P \in \Psi^c(M, \emptyset), Q \in \Psi^c(N, \emptyset)\) and \(p' = 2.0 \ldots 2.1.0\). Then:

- \(P_0 = c^n((\lambda x.(\lambda y.P[y := c(cy)])x)Q) \overrightarrow{\beta \eta} c^n((\lambda y.P[y := c(cy)])Q)\)
- \(|\langle P_0, p' \rangle|^c = |\langle P_0, 2^n.1.0 \rangle|^c = p\)
- \(c^n((\lambda y.P[y := c(cy)])Q) \in \Psi^c((\lambda y.M)N, \{0\})\)
An Extension of Koletsos and Stavrinos’s method [KS08]

$\beta\eta$-developments

Let $c \notin \text{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$.

- Let $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta\eta} M'$. We call the unique $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta\eta}$, such that for all $N \in \Psi^c(M, \mathcal{F})$ there exist $N' \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}_{N}^{\beta\eta}$ such that $N \xrightarrow{p'}_{\beta\eta} N'$ and $|\langle N, p' \rangle|^c = p$, the set of $\beta\eta$-residuals of $\mathcal{F}$ in $M'$ relative to $p$.

- A one-step $\beta\eta$-development of $\langle M, \mathcal{F} \rangle$, denoted $\langle M, \mathcal{F} \rangle \rightarrow_{\beta\eta d} \langle M', \mathcal{F}' \rangle$, is a $\beta\eta$-reduction $M \xrightarrow{p}_{\beta\eta} M'$ where $p \in \mathcal{F}$ and $\mathcal{F}'$ is the set of $\beta\eta$-residuals of $\mathcal{F}$ in $M'$ relative to $p$. A $\beta\eta$-development is the transitive closure of a one-step $\beta\eta$-development. We write $M \rightarrow_1 M'$ for the $\beta\eta$-development $\langle M, \mathcal{F} \rangle \rightarrow^{*}_{\beta\eta d} \langle M', \mathcal{F}' \rangle$.

Lemma

If $c \notin \text{fv}(M)$, $M \rightarrow_1 M_1$ and $M \rightarrow_1 M_2$ then there exists $M'$ such that $M_1 \rightarrow_1 M_3$ and $M_2 \rightarrow_1 M_3$. 
The transitive reflexive closure of $\rightarrow_{\beta \eta}$ is equal to the transitive reflexive closure of $\rightarrow_1$. We are now able to prove the (non-strict) inclusion of $\Lambda$ in CRBE and the equality between these sets:

**Lemma**

$c \notin \text{fv}(M) \Rightarrow M \in \text{CRBE}$. 
J. Gallier.
Typing untyped λ-terms, or reducibility strikes again!.

S. Ghilezan and S. Likavec.
Reducibility: A ubiquitous method in lambda calculus with
intersection types.

G. Koletsos and G. Stavrinos.
Church-Rosser property and intersection types.
To appear.