Beyond $\beta$-Reduction in Church’s $\lambda\rightarrow$ *

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Abstract

In this paper, we shall write $\lambda_\to$ using a notation, item notation, which enables one to make more redexes visible, and shall extend $\beta$-reduction to all visible redexes. We will prove that $\lambda_\to$ written in item notation and accommodated with extended reduction, satisfies all its original properties (such as Church Rosser, Subject Reduction and Strong Normalisation). The notation itself is very simple: if $I$ translates classical terms to our notation, then $I (t_1 t_2) \equiv (I (t_2) \delta) I (t_1)$ and $I (\lambda_\eta \cdot t) \equiv (\rho \lambda \eta) I (t)$. For example, $t \equiv ((\lambda_{x_1} x_4) (\lambda_{x_2} x_3) \lambda_{x_5 . x_1 \to x_2 . x_3 x_4} x_5) x_2 x_1$, can be written in our item notation as $I (t) \equiv (y_1 \delta) (y_2 \delta) (x_4 \lambda_{x_1} \lambda_{x_2} \lambda_{x_3} \lambda_{x_4}) ((X_1 \to X_2) \lambda_{x_5}) (y_4 \delta) x_5$ where the visible redexes are based on all the matching $\delta \lambda$-couples. So here, the redexes are based on $(y_2 \delta) (X_4 \lambda_{x_1})$, $(x_2 \delta) (X_3 \lambda_{x_2})$ and $(y_1 \delta) ((X_1 \to X_2) \lambda_{x_5})$. In classical notation however, only the redexes based on $(\lambda_{x_1} x_4, \cdots) x_2$ and $(\lambda_{x_5} x_1, \cdots) x_3$ are immediately visible. The third redex, $(\lambda_{x_1} x_4, \cdots) x_2$, only becomes visible when the first two redexes have been contracted. We extend $\beta$-reduction so that we can contract newly visible redexes even before other redexes have been contracted. So in our example above, $(y_1 \delta) ((X_1 \to X_2) \lambda_{x_5})$ can be contracted before $(y_2 \delta) (X_4 \lambda_{x_1})$ or $(x_2 \delta) (X_3 \lambda_{x_2})$. This refinement (which cannot be done in classical notation) is achieved by generalising the axiom $\beta$ from $(t_1 \delta) (\rho \lambda \eta) t_2 \to_\beta t_2 [v := t_1]$ to $(t_1 \delta) (\rho \lambda \eta) y_2 \to_\beta y_2 \pi (t_2 [v := t_1])$ for $\pi$ consisting of matching $\delta \lambda$-couples only. Hence, as $(x_2 \delta) (X_4 \lambda_{x_1}) (x_3 \delta) (X_3 \lambda_{x_2})$ consists of matching $\delta \lambda$-couples, we get that $I (t) \sim_\beta (x_2 \delta) (X_4 \lambda_{x_1} \lambda_{x_2}) (x_3 \delta) (X_3 \lambda_{x_2}) ((x_4 \delta) x_5) [x_5 := x_1]$. Furthermore, with our item notation, it is possible to refine reduction by rewriting (or reshuffling) terms so that matching $\delta \lambda$-couples occur adjacent to each other. For example, we can rewrite $I (t)$ above as $(x_2 \delta) (X_4 \lambda_{x_1}) (x_3 \delta) (X_3 \lambda_{x_2}) (x_4 \delta) ((X_1 \to X_2) \lambda_{x_5}) (x_4 \delta) x_5$. We shall formalise term reshuffling and shall show that it is correct and preserves both $\beta$-reduction and typing.

Keywords: $\beta$-reduction, Church Rosser, Subject Reduction, Strong Normalisation.
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1 Introduction

The notation of this paper, item notation, is a novel notation where the argument is given before the function, the type is given before the lambda, and where the parentheses are grouped differently than those of the classical notation. So that, if \( I \) translates classical terms into our notation, then \( I(t_1 t_2) \) is written as \((I(t_2)\delta)I(t_1)\) and \( I(\lambda_{\nu}, t) \) is written as \((\rho\lambda_{\nu})I(t)\). Both \((t\delta)\) and \((\rho\lambda_{\nu})\) are called items.

Example 1.1 \( I((\lambda_{x_6}:X_1 \rightarrow (X_2 \rightarrow X_3), \lambda_{y}:X_1 \rightarrow x y)z) \equiv ((z\delta)(X_1 \rightarrow (X_2 \rightarrow X_3)\lambda_{x})(X_1\lambda_{y})(y\delta)x) \). The items are \((z\delta), (X_1 \rightarrow (X_2 \rightarrow X_3)\lambda_{x}), (X_1\lambda_{y}), \) and \((y\delta)\).

Before we discuss the calculus and the properties of typing, let us see why we want to extend the notion of a redex and to refine \( \beta \)-reduction.

In the classical \( \lambda \)-calculus, the notions of redex, and of \( \beta \)-reduction are described as follows:

Definition 1.2 (Redexes and \( \beta \)-reduction in classical notation)

In the classical notation of the \( \lambda \)-calculus, a redex is of the form \((\lambda_{v}, t)t'\). Moreover, one-step \( \beta \)-reduction \( \rightarrow_{\beta} \) is the compatible relation generated out of the axiom \( \beta: (\lambda_{v}, t)t' \rightarrow \beta t[v := t'] \).

Many step \( \beta \)-reduction \( \rightarrow_{\beta} \), is the reflexive transitive closure of \( \rightarrow_{\beta} \).

With our item notation, classical redexes and \( \beta \)-reduction take the following form:

Definition 1.3 (Classical redexes and \( \beta \)-reduction in item notation)

In the item notation of the \( \lambda \)-calculus, a redex is of the form \((t'\delta)(\rho\lambda_{v})\). We call the pair \((t'\delta)(\rho\lambda_{v})\), a \( \delta \lambda \)-pair, or a \( \delta \lambda \)-segment. The classical \( \beta \)-reduction axiom is: \((t'\delta)(\rho\lambda_{v}) \rightarrow_{\beta} t'[v := t']\). One and many step \( \beta \)-reduction are defined as in Definition 1.2.

Example 1.4 In the classical term \( t \equiv ((\lambda_{x_7}:X_4 \rightarrow \lambda_{x_6}:X_3 \rightarrow \lambda_{x_5}:X_1 \rightarrow X_2 \rightarrow x_5 x_4)x_3)x_2 \) \( x_1 \), we have the following redexes (the fact that neither \( x_6 \) nor \( x_7 \) appear as free variables in their respective scopes does not matter here; this is just to keep the example simple and clear):

1. \((\lambda_{x_6}:X_3 \rightarrow \lambda_{x_5}:X_1 \rightarrow X_2 \rightarrow x_5 x_4)x_3 \)
2. \((\lambda_{x_7}:X_4 \rightarrow \lambda_{x_6}:X_3 \rightarrow \lambda_{x_5}:X_1 \rightarrow X_2 \rightarrow x_5 x_4)x_3 \) \( x_2 \)

In item notation \( t \) becomes \((x_1 \delta)(x_2 \delta)(X_1\lambda_{x_7})(x_3 \delta)(X_3\lambda_{x_6})(X_1 \rightarrow X_2)(X_3\lambda_{x_5})(x_4 \delta)x_5 \). Here, the two classical redexes correspond to \( \delta \lambda \)-pairs as follows:

1. \((\lambda_{x_6}:X_3 \rightarrow \lambda_{x_5}:X_1 \rightarrow X_2 \rightarrow x_5 x_4)x_3 \) corresponds to \((x_2 \delta)(X_3\lambda_{x_6})\). \( ((X_1 \rightarrow X_2)(X_3\lambda_{x_5})(x_4 \delta)x_5 \) is ignored as it is easily retrievable in item notation. It is the maximal subterm of \( t \) to the right of \((X_3\lambda_{x_6})\).

2. \((\lambda_{x_7}:X_4 \rightarrow \lambda_{x_6}:X_3 \rightarrow \lambda_{x_5}:X_1 \rightarrow X_2 \rightarrow x_5 x_4)x_3 \) \( x_2 \) corresponds to \((x_2 \delta)(X_4\lambda_{x_7})\).

Again \((x_3 \delta)(X_3\lambda_{x_6})((X_1 \rightarrow X_2)(X_3\lambda_{x_5})(x_4 \delta)x_5 \) is ignored for the same reason as above.

There is however a third redex which is not immediately visible in the classical term; namely, \((\lambda_{x_5}:X_1 \rightarrow X_2 \rightarrow x_5 x_4)x_1 \). Such a redex will only be visible after we have contracted the above two redexes (we will not discuss the order here). In fact, assume we contract the second redex in the first step, and the first redex in the second step. I.e.

\[
((\lambda_{x_7}:X_4 \rightarrow \lambda_{x_6}:X_3 \rightarrow \lambda_{x_5}:X_1 \rightarrow X_2 \rightarrow x_5 x_4)x_3)x_2 \xrightarrow{\beta} x_1 \rightarrow \beta
((\lambda_{x_6}:X_3 \rightarrow \lambda_{x_5}:X_1 \rightarrow X_2 \rightarrow x_5 x_4)x_3)x_1 \xrightarrow{\beta} \beta
(\lambda_{x_5}:X_1 \rightarrow X_2 \rightarrow x_5 x_4)x_1 \rightarrow \beta x_1 x_4
\]
Now, even though all these three redexes are needed in order to get the normal form of $t$, only the first two were visible in the classical term at first sight. The third could only be seen once we had contracted the first two redexes. In item notation, the third redex $(\lambda x_2.x_1 \rightarrow x_2 x_3 x_4) x_1$ is visible as it corresponds to the matching $(x_1 \delta)((X_1 \rightarrow X_2) \lambda x_2)$ where $(x_1 \delta)$ and $((X_1 \rightarrow X_2) \lambda x_2)$ are separated by the segment $(x_2 \delta)(X_1 \lambda x_1)(x_3 \delta)(X_3 \lambda x_6)$. Hence, by extending the notion of a redex from being a $\delta$-item adjacent to a $\lambda$-item, to being a matching pair of $\delta$- and $\lambda$-items, we can make more redexes visible. This extension furthermore is simple, as in $(t_1 \delta)s(\rho \lambda v)$, we say that $(t_1 \delta)$ and $(\rho \lambda v)$ match if $s$ has the same structure as a matching composite of opening and closing brackets, each $\delta$-item corresponding to an opening bracket and each $\lambda$-item corresponding to a closing bracket. For example, in $t$ above, $(x_1 \delta)$ and $((X_1 \rightarrow X_2) \lambda x_2)$ match as $(x_2 \delta)(X_1 \lambda x_1)(x_3 \delta)(X_3 \lambda x_6)$ has the bracketing structure $[ ]$ (see Figure 1 which is drawn ignoring types just for the sake of argument). With this

\begin{figure}[h]
\centering
\begin{tikzpicture}
    \node (x1d) at (0,0) {$(x_1 \delta)$};
    \node (x2d) at (1,0) {$(x_2 \delta)$};
    \node (lambda2) at (2,0) {$(\lambda x_2)$};
    \node (x3d) at (3,0) {$(x_3 \delta)$};
    \node (lambda6) at (4,0) {$(\lambda x_6)$};
    \node (lambda5) at (5,0) {$(\lambda x_5)$};
    \node (x4d) at (6,0) {$(x_4 \delta)$};
    \node (x5) at (7,0) {$x_5$};
    \draw (x1d) -- (x2d);
    \draw (x2d) -- (lambda2);
    \draw (lambda2) -- (x3d);
    \draw (x3d) -- (lambda6);
    \draw (lambda6) -- (lambda5);
    \draw (lambda5) -- (x4d);
\end{tikzpicture}
\caption{Redexes in item notation}
\end{figure}

extension of redexes, we refine $\beta$-reduction in two different ways:

1. By changing $\beta$ from $(t_1 \delta)(\rho \lambda v)t_2 \rightarrow_\beta t_2[v := t_1]$ to $(t_1 \delta)s(\rho \lambda v)t_2 \rightsquigarrow_\beta s(t_2[v := t_1])$ if $(t_1 \delta)$ and $(\rho \lambda v)$ match.

2. By reshuffling terms so that matching $\delta$’s and $\lambda$’s occur adjacently.

We start by showing that $\rightsquigarrow_\beta$ (the reflexive transitive closure of $\rightsquigarrow_\beta$) is a generalisation of $\rightarrow_\beta$ (Lemma 3.7). We will then show that $\lambda_\rightarrow$ with $\rightsquigarrow_\beta$ satisfies all the desirable typing properties. In particular, we will establish that $\lambda_\rightarrow$ extended with $\rightsquigarrow_\beta$ satisfies the following:

1. Church Rosser: this says that if a program is evaluated in two different ways, then the answer stays the same (Theorem 3.10).

2. Subject Reduction: this says that if a program $P$ is well-typed then the program obtained from evaluating some steps in $P$ is also well-typed (Theorem 3.13).

3. Unicity of Types: this says that a well-typed program has a unique type and that two equal programs have the same type (Lemma 3.15).

4. Strong Normalisation: this says that all ways of evaluating a well-typed program terminate (Theorem 3.21).

We will furthermore show that term reshuffling is correct. In particular, we shall show that $\lambda_\rightarrow$ accommodated with term reshuffling $TS$, satisfies the following:

1. Reshuffling a term, moves all $\delta$’s next to their matching $\lambda$’s (Lemma 4.9).
2. Reshuffling terms preserves $\rightarrow_\beta$. That is, if $t \sim_\beta t'$ then there exists $t''$ such that $TS(t) \rightarrow_\beta t''$ and $TS(t') \equiv TS(t'')$ (Lemma 4.11).

3. Reshuffling terms preserves types. That is, if $\Gamma \vdash t : \rho$ then $\Gamma \vdash TS(t) : \rho$ (Lemma 4.13).

2 $\lambda_\rightarrow$ in item notation

In this section, we shall introduce the known $\lambda_\rightarrow$ (which uses the ordinary $\beta$-reduction $\rightarrow_\beta$), and its properties. We shall write $\lambda_\rightarrow$ immediately in item notation. That is, we assume a translation function $I$ from terms in classical notation to terms in item notation such that:

$$
I(v) = v \quad \text{if } v \text{ is a variable} \\
I(\lambda_v.\rho t) = (\rho \lambda_v)I(t) \\
I(t_1 t_2) = (I(t_2) \delta)I(t_1)
$$

2.1 The basic theory

In Church's system $\lambda_\rightarrow$, types and terms are defined as follows:

**Definition 2.1 (Types of $\lambda_\rightarrow$)**
The set of types $\mathcal{T}$ of $\lambda_\rightarrow$ is defined as follows:

$$
\mathcal{T} ::= \mathcal{V} \mid (\mathcal{T} \rightarrow \mathcal{T}) \quad \text{Types} \\
\mathcal{V} ::= \{X_0, X_1, \cdots\} \quad \text{Type variables}
$$

That is, types are either variables such as $X_0, X_1, X_2, \ldots$ or arrow types.

**Definition 2.2 (Terms of $\lambda_\rightarrow$)**
The set of terms $\Lambda_T$ of $\lambda_\rightarrow$ is defined as follows:

$$
\Lambda_T ::= \mathcal{V} \mid (\mathcal{T} \lambda_V)\Lambda_T \mid (\Lambda_T \delta)\Lambda_T \quad \text{Terms} \\
\mathcal{V} ::= \{x_0, x_1, \cdots\} \quad \text{Variables}
$$

In other words, a term is either a variable $x_0, x_1, x_2, \ldots$, or an abstraction or an application.

**Notation 2.3** We let $\rho, \rho', \rho_1, \rho_2, \ldots$ range over types, $\alpha, \alpha', \alpha_1, \alpha_2, \ldots$ range over type variables. Furthermore, we take $t, t', t_1, t_2, \ldots$ to range over terms and let $v, v', v_1, \ldots$ range over variables.

Parentheses moreover will be omitted when no confusion occurs.

We understand $\rho_1 \rightarrow \rho_2 \rightarrow \cdots \rightarrow \rho_n$ to mean $(\rho_1 \rightarrow (\rho_2 \rightarrow \cdots \rightarrow (\rho_{n-1} \rightarrow \rho_n) \cdots))$.

Bound and free variables in $\lambda_\rightarrow$ are defined as usual. We write $BV(t)$ and $FV(t)$ to represent the bound and free variables of $t$ respectively. Substitution moreover, is defined in the usual way. Furthermore, we take terms to be equivalent up to variable renaming. For example, we take $(\rho \lambda_{x_0})x_0 \equiv (\rho \lambda_{x_1})x_1$. We assume moreover, the Barendregt variable convention which is formally stated as follows:

**Definition 2.4 (BC: Barendregt's Convention for $\lambda_\rightarrow$)**
Names of bound variables will always be chosen such that they differ from the free ones in a term. Hence, we will not have $(v \delta)(\rho \lambda_v)v$, but $(v \delta)(\rho \lambda_{v'})v'$ instead.
Definition 2.5 (Compatibility)
We say that a relation $\to$ on terms is compatible iff the following holds:

$$
\begin{align*}
  t \to t' & \quad (t\delta)t_1 \to (t'\delta)t_1 \\
  t \to t' & \quad (t_1\delta)t \to (t_1\delta)t' \\
  t \to t' & \quad (\rho\lambda_v)t \to (\rho\lambda_v)t'
\end{align*}
$$

Basically compatibility means that if $t \to t'$ then $T[t] \to T[t']$ where $T[\_]$ is a “term with a hole in it”.

Definition 2.6 ($\beta$-reduction $\to_\beta$ in $\lambda \to$)
In $\lambda \to$, $\beta$-reduction $\to_\beta$, is the least compatible relation generated out of the following axiom:

$$(\beta) \quad \quad (t_1\delta)(\rho\lambda_v)t \to_\beta t[v := t_1]$$

We take $\to_\beta$ to be the reflexive transitive closure of $\to_\beta$ and $=_\beta$ to be the least equivalence relation generated by $\to_\beta$.

Definition 2.7 ((main) items, (main, $\delta\lambda$-)segments, context, body, endvar, weight)

- If $v$ is a variable, $\rho$ is a type and $t$ is a term then $(\rho\lambda_v)$ and $(t\delta)$ are items (the first is called $\lambda$-item, the second $\delta$-item). We use $s, s_1, s_i, \ldots$ to range over items.

- A concatenation of zero or more items is a segment. We use $\pi, \pi_1, \pi_i, \ldots$ as metavariables for segments. We write $\emptyset$ for the empty segment.

- Each term $t$ is the concatenation of zero or more items and a variable: $t \equiv s_1s_2\cdots s_nv$. These items $s_1, s_2, \ldots, s_n$ are called the main items of $t$.

- Analogously, a segment $\pi$ is a concatenation of zero or more items: $\pi \equiv s_1s_2\cdots s_n$; again, these items $s_1, s_2, \ldots, s_n$ (if any) are called the main items, this time of $\pi$.

- A concatenation of adjacent main items (in $t$ or $\pi$), $s_m \cdots s_{m+k}$, is called a main segment (in $t$ or $\pi$).

- A context is a segment which consists of only $\lambda$-items. We use $\Gamma, \Gamma', \Gamma_1, \Gamma_2, \ldots$ to range over contexts.

- A $\delta\lambda$-segment is a $\delta$-item immediately followed by a $\lambda$-item.

- Let $t \equiv \pi v$ be a term. Then we call $\pi$ the body of $t$, denoted $\text{body}(t)$, and $v$ the end variable of $t$, or $\text{endvar}(t)$. It follows that $t \equiv \text{body}(t) \text{ endvar}(t)$.

- The weight of a $\lambda \to$-segment $\pi$, $\text{weight}(\pi)$, is the number of main items that compose the segment. Moreover, we define $\text{weight}(t) = \text{weight}(\text{body}(t))$.

Definition 2.8 (Statements)
A statement is of the form $t : \rho$, $t$ and $\rho$ are called the subject and the type of the statement respectively.
Convention 2.9 In a context, we never have two occurrences of \( \lambda_v \) (for the same \( v \)). Hence, contexts are what [Barendregt 92] calls bases.

We need the following definition over contexts:

Definition 2.10 Let \( \Gamma = (\rho_1 \lambda_v_1)(\rho_2 \lambda_v_2) \cdots (\rho_k \lambda_v_k) \) be a context. Then

1. \( \text{dom}(\Gamma) = \{v_1, v_2, \ldots, v_k\} \)
2. \( (\rho \lambda_v) \in' \Gamma \) if \( (\rho \lambda_v) \) is an item of \( \Gamma \). If \( \Gamma' \) is a context such that all items of \( \Gamma' \) are also items of \( \Gamma \), we write \( \Gamma' \subseteq \Gamma \).
3. Let \( V_0 \) be a set of term variables. \( \Gamma | V_0 \) (the restriction of \( \Gamma \) to \( V_0 \)) is the context which only contains the items \( (\rho \lambda_v) \in' \Gamma \) for which \( v \in V_0 \), in the original order.

Now for the formulation of the typing rules we can use the following definitions for the derivation of so-called judgements of the form \( \Gamma \vdash t : \rho \).

Definition 2.11 (Typing rules of \( \lambda \rightarrow \))
A statement \( t : \rho \) is derivable in the context \( \Gamma \), notation \( \Gamma \vdash t : \rho \), if \( t : \rho \) can be derived using the following rules:

\[
\begin{align*}
(Axiom) & \quad \Gamma \vdash v : \rho \quad \text{if } (\rho \lambda_v) \in' \Gamma \\
(\rightarrow -elimination) & \quad \Gamma \vdash t : \rho \quad \Gamma \vdash t' : (\rho \rightarrow \rho') \\
(\rightarrow -introduction) & \quad \Gamma, (\rho \lambda_v) \vdash t : \rho' \\
& \quad \Gamma \vdash (\rho \lambda_v)t : (\rho \rightarrow \rho')
\end{align*}
\]

2.2 Properties of \( \lambda \rightarrow \)
Here we list the properties of \( \lambda \rightarrow \) (that we will establish for extended reduction) without proofs. The reader can refer to [Barendregt 92] for details.

Theorem 2.12 (The Church Rosser Theorem)
If \( t \rightarrow_\beta t_1 \) and \( t \rightarrow_\beta t_2 \) then there exists \( t_3 \) such that \( t_1 \rightarrow_\beta t_3 \) and \( t_2 \rightarrow_\beta t_3 \)

Lemma 2.13 (Context lemma)
1. \( \forall \Gamma \forall \delta \forall \rho \left[ \Gamma \subseteq' \Gamma' \land \Gamma \vdash t : \rho \Rightarrow \Gamma' \vdash t : \rho \right] \)
2. \( \forall \Gamma \forall \delta \forall \rho \left[ \Gamma \vdash t : \rho \Rightarrow \text{FV}(t) \subseteq \text{dom}(\Gamma) \right] \)
3. \( \forall \Gamma \forall \delta \forall \rho \left[ \Gamma \vdash t : \rho \Rightarrow \Gamma \left[ \text{FV}(t) \vdash t : \rho \right] \right] \)

Proof: All by induction on the derivation \( \Gamma \vdash t : \rho \).

Lemma 2.14 (Generation lemma)
1. \( \forall \Gamma \forall v \forall \rho \left[ \Gamma \vdash v : \rho \Rightarrow (\rho \lambda_v) \in' \Gamma \right] \)
2. \( \forall \Gamma \forall \lambda \forall \rho \left[ \Gamma \vdash (t' \lambda) : \rho \Rightarrow \exists \rho' \left[ \Gamma \vdash t : (\rho \rightarrow \rho') \land \Gamma \vdash t' : \rho' \right] \right] \)
3. $\forall \Gamma \forall \alpha \forall \rho. \forall \rho', \exists \rho'' : [\rho \equiv \rho' \land \rho'' \vdash \Gamma \vdash t : \rho]$

Proof: By induction on the derivation of $\Gamma \vdash t : \rho$.

Lemma 2.15 (Subterm lemma)
\[ \forall \Gamma \forall t \forall \rho. \forall \rho' \exists \rho'' : [\Gamma \vdash t : \rho \land \rho' \equiv \rho'' \vdash \Gamma \vdash t : \rho'] \]

Proof: By induction on $t$.

Lemma 2.16 (Substitution lemma)
1. $\forall \Gamma \forall t \forall \rho. \forall \alpha \in V. [\Gamma \vdash t : \rho \equiv \rho' \vdash t[\alpha \leftarrow \rho'] : \rho[\alpha \leftarrow \rho']$]
2. $\forall \Gamma \forall t \forall \rho. [\Gamma \vdash t : \rho' \land \rho \equiv \rho' \vdash \Gamma \vdash t : \rho']$

Proof: 1: by induction on $\Gamma \vdash t : \rho$. 2: by induction on $\Gamma \vdash t : \rho'$.

Theorem 2.17 (Subject Reduction)
\[ \forall \Gamma \forall t \forall \rho. \forall \beta'. [\Gamma \vdash t : \rho \equiv \rho' \vdash \Gamma \vdash t' : \rho'] \]

Proof: By induction on $\Gamma \vdash t : \rho$, case 1.

Lemma 2.18 (Unicity of Types)
1. $\forall \Gamma \forall t \forall \rho. [\Gamma \vdash t : \rho \land \rho \equiv \rho' \vdash \Gamma \vdash t : \rho']$
2. $\forall \Gamma \forall t \forall \rho. [\Gamma \vdash t : \rho \land \rho \equiv \rho'$]

Proof: 1 is by an easy induction on $t$. 2 is by Church Rosser, Subject Reduction and 1.

Definition 2.19 (Strongly Normalising terms with respect to $\Rightarrow_{\beta}$)
We say that a term $t$ is strongly normalising with respect to $\Rightarrow_{\beta}$ iff every reduction path using $\Rightarrow_{\beta}$ and starting at $t$ terminates.

Theorem 2.20 (Strong Normalisation with respect to $\Rightarrow_{\beta}$)
If $\Gamma \vdash t : \rho$ then $t$ is strongly normalising with respect to $\Rightarrow_{\beta}$.

3 Generalising reduction

In this section we shall extend the classical notions of redexes and $\beta$-reduction of $\lambda \rightarrow$. We shall show that this extension of $\lambda \rightarrow$ satisfies all the listed properties in Section 2.

3.1 Extending redexes and $\beta$-reduction
When one desires to start a $\beta$-reduction on the basis of a certain $\delta$-item and a $\lambda$-item occurring in one segment, the matching of the $\delta$ and the $\lambda$ in question is the important thing, even when the $\delta$- and $\lambda$-items are separated by other items. I.e., the relevant question is whether they may together become a $\delta \lambda$-segment after a number of $\beta$-steps. This depends solely on the structure of the intermediate segment. If such an intermediate segment is well-balanced then the $\delta$-item and the $\lambda$-item match and $\beta$-reduction based on these two items may take place. Here is the definition of well-balanced segments:
Definition 3.1 (well-balanced segments in $\lambda\rightarrow$)

- The empty segment $\emptyset$ is a well-balanced segment;
- If $s$ is a well-balanced segment, then $(t')s(t\rho)$ is a well-balanced segment.
- The concatenation of well-balanced segments is a well-balanced segment;

A well-balanced segment has the same structure as a matching composite of opening and closing brackets, each $\delta$- (or $\lambda$-)item corresponding with an opening (resp. closing) bracket.

Now we can easily define what matching $\delta\lambda$-couples are, given a segment $\pi$. Namely, they are a main $\delta$-item and a main $\lambda$-item separated by a well-balanced segment. Such couples are reducible couples. The $\delta$-item and $\lambda$-item of the $\delta\lambda$-couple are said to match and each of them is called a partner or a partnered item. The items in a segment that are not partnered are called bachelor items. The following definition summarizes all this:

Definition 3.2 (match, $\delta\lambda$- or reducible couple, partner, partnered item, bachelor item, bachelor segment)

Let $t$ be a $\lambda\rightarrow$-term. Let $\pi \equiv s_1 \ldots s_n$ be a segment occurring in $t$.

- We say that $s_i$ and $s_j$ match, when $1 \leq i < j \leq n$, $s_i$ is a $\delta$-item, $s_j$ is a $\lambda$-item, and the sequence $s_{i+1}, \ldots, s_{j-1}$ forms a well-balanced segment.
- When $s_i$ and $s_j$ match, we call $s_i s_j$ a $\delta\lambda$-couple or reducible couple.
- When $s_i$ and $s_j$ match, we call both $s_i$ and $s_j$ the partners in the $\delta\lambda$-couple. We also say that $s_i$ and $s_j$ are partnered items.
- All $\delta$- (or $\lambda$-)items $s_k$ in $t$ that are not partnered, are called bachelor $\lambda$- (resp. $\delta$-)items.
- A segment consisting of bachelor items only, is called a bachelor segment.
- The segment $s_{i_1} \ldots s_{i_m}$ consisting of all bachelor main $\lambda$- (or $\delta$-)items of $\pi$ is called the bachelor $\lambda$- (or $\delta$-)segment of $\pi$.

Example 3.3 In $\pi \equiv (\rho_1 \lambda v_1)(\rho_2 \lambda v_2)(t_1 \delta)(\rho_3 \lambda v_3)(\rho_4 \lambda v_4)(t_2 \delta)(t_3 \delta)(t_4 \delta)(\rho_5 \lambda v_5)(\rho_6 \lambda v_6)(t_5 \delta)$:

- $(t_1 \delta)$ matches with $(\rho_3 \lambda v_3)$, $(t_4 \delta)$ matches with $(\rho_5 \lambda v_5)$ and $(t_5 \delta)$ with $(\rho_6 \lambda v_6)$. The segments $(t_1 \delta)(\rho_3 \lambda v_3)$ and $(t_4 \delta)(\rho_5 \lambda v_5)$ are $\delta\lambda$-segments (and $\delta\lambda$-couples). There is another $\delta\lambda$-couple in $\pi$, viz. the couple of $(t_3 \delta)$ and $(\rho_6 \lambda v_6)$.
- $(t_1 \delta), (\rho_3 \lambda v_3), (t_3 \delta), (t_4 \delta), (\rho_5 \lambda v_5)$ and $(\rho_6 \lambda v_6)$, are the partnered main items of $\pi$, $(\rho_1 \lambda v_1), (\rho_2 \lambda v_2), (\rho_4 \lambda v_4), (t_2 \delta)$ and $(t_5 \delta)$, are bachelor items.
- $(\rho_1 \lambda v_1)(\rho_2 \lambda v_2)$ and $(\rho_4 \lambda v_4)(t_2 \delta)$ are bachelor segments, whereas $(t_3 \delta)(t_4 \delta)(\rho_5 \lambda v_5)$ and $(t_3 \delta)(t_4 \delta)(\rho_6 \lambda v_6)$ are non-bachelor, the latter also being a well-balanced segment.

De Bruijn uses another terminology; see e.g. [de Bruijn 93]. In his phrasing, $\delta$-items are applicators or $A$'s, and $\lambda$-items are abstractors or $T$'s. For $\delta\lambda$-segments he uses the word $\text{AT-pair}$ and for $\delta\lambda$-couples he uses $\text{AT-couples}$. Void $\beta$-reduction (i.e.: the reduction $(t_1 \delta)(\rho \lambda v)t \rightarrow_\beta t$ if $v \not\in FV(t)$), he calls $\text{AT-removal}$. 

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Having argued above that $\beta$-reduction should not be restricted to the $\delta\lambda$-segments but may take into account other candidates, we can extend our notion of $\beta$-reduction in this vein. That is to say, we may allow $\delta\lambda$-couples to have the same “reduction rights” as $\delta\lambda$-segments. In order to accomplish this, we change the $\beta$-reduction of Definition 2.6 to the following

**Definition 3.4** (General $\beta$-reduction $\leadsto_\beta$ for $\lambda_\to$)
General one-step $\beta$-reduction $\leadsto_\beta$, is the least compatible relation generated out of the following axiom:

$$(t_1\delta)(\rho_\lambda \tau_\delta) t \leadsto_\beta \tau(t[v := t_1])$$ if $\tau$ is well-balanced

General $\leadsto_\beta$ is the reflexive and transitive closure of $\leadsto_\beta$ and $\sim_\beta$ is the least equivalence relation generated by $\leadsto_\beta$.

**Example 3.5** Take Example 1.4. As $(x_2\delta)(x_1\lambda x_1)(x_3\delta)(x_3\lambda x_2)$ is a well-balanced segment, then $(x_1\delta)((X_1 \to X_2)\lambda x_2)$ is a $\delta\lambda$-couple and

$$t \equiv (x_1\delta)(x_2\delta)(x_3\delta)(x_4\delta)(X_1 \to X_2)\lambda x_2)(x_4\delta) x_5 \leadsto_\beta$$

$$((x_2\delta)(x_3\delta)(x_3\lambda x_2)((x_4\delta)x_5[x_5 := x_1])$$

The reducible couple $(x_1\delta)((X_1 \to X_2)\lambda x_2)$ also has a corresponding ("generalised") redex in the traditional notation, which will appear after two one-step $\beta$-reductions, leading to $(\lambda_{x_3}:X_1 \to (x_2\cdot x_5:x_1) x_1$. With $\leadsto_\beta$, we could reduce $((\lambda_{x_7}:X_4, \lambda_{x_6}:X_5, \lambda_{x_5}:X_1 \to X_2, x_5:x_4) x_3) x_2$ to $(\lambda_{x_7}:X_4, \lambda_{x_6}:X_5, x_1 x_4) x_3 x_2$. This reduction is difficult to carry out in the classical $\lambda$-calculus.

The item notation enables a new and important sort of reduction which has not yet been studied in relation to the standard $\lambda$-calculus up to date. We believe that this generalised reduction (introduced in [Nederpelt 73]) can only be obtained tidily in a system formulated using some form of our item notation. In fact, one is to compare the bracketing structure of the classical term $t$ of Example 1.4, with the bracketing structure of the corresponding term in item notation:

**Example 3.6** The "bracketing structure" of $((\lambda_{x_7}:X_4, \lambda_{x_6}:X_5, \lambda_{x_5}:X_1 \to (\cdots) x_3) x_2) x_1$ is compatible with $[[[i]]j]]'$, where $'i'$ and $'j'$ match. In item notation however, it has the bracketing structure $[[[i]]]$. We strongly believe that it is the item notation which enables us to extend reduction smoothly beyond $\to_\beta$. Because a well-balanced segment may be empty, the general $\beta$-reduction rule presented above is really an extension of the classical $\beta$-reduction rule.

**Lemma 3.7** Let $t_1, t_2$ be $\lambda_\to$-terms. If $t_1 \to_\beta t_2$ in the sense of Definition 2.6, then $t_1 \leadsto_\beta t_2$ in the sense of Definition 3.4. Moreover, if $t_1 \leadsto_\beta t_2$ comes from contracting a $\delta\lambda$-segment then $t_1 \to_\beta t_2$.

**Proof:** Obvious as a $\delta\lambda$-segment is a $\delta\lambda$-couple.

### 3.2 Properties of $\lambda_\to$ with generalised reduction

If we look at Section 2.2, we see that Context, Generation, Subterm, Substitution and Unicity of Types (part 1) lemmas are not affected by our extension of Reduction. Hence, they all still hold for $\lambda_\to$ with $\sim_\beta$. The only three (and very important) properties that get affected
by $\rightsquigarrow_\beta$ are: Church Rosser (Theorem 2.12), Subject Reduction (Theorem 2.17), Unicity of Types (part 2, Lemma 2.18), and Strong Normalisation (Theorem 2.20). In this section, we shall show that these properties are preserved for $\rightsquigarrow_\beta$.

The proof of the generalised Church Rosser theorem is simple. The idea is to show that if $t \rightsquigarrow_\beta t'$ then $t =_\beta t'$ and to use the Church Rosser property for $=_\beta$.

**Lemma 3.8** If $t \rightsquigarrow_\beta t'$ then $t =_\beta t'$.

**Proof:** It suffices to consider the case $t \equiv \overline{s_1}(t_1 \delta) \overline{s}(\rho \lambda v) t_2$ where the contracted redex is based on $(t_1 \delta)(\rho \lambda v)$, $t' \equiv \overline{s_1} \overline{s}(t_2[v := t_1])$, and $\overline{s}$ is well-balanced (hence weight(\overline{s}) is even). We shall prove the lemma by induction on weight(\overline{s}).

- Case weight(\overline{s}) = 0 then obvious as $\rightsquigarrow_\beta$ coincides with $\rightarrow_\beta$ in this case.

- Assume the property holds when weight(\overline{s}) = 2n. Take $\overline{s}$ such that weight(\overline{s}) = 2n + 2. Now, $\overline{s} \equiv (t_3 \delta) \overline{s'}(\beta \lambda v') \overline{s''}$ where $\overline{s'}$, $\overline{s''}$ are well-balanced. Assume $v \neq v'$ (if necessary, use renaming).

  - As $\overline{s}(t_2[v := t_1]) \rightsquigarrow_\beta \overline{s'}(\overline{s''}(t_2[v := t_1][v' := t_3]))$, we get by IH and compatibility that $t' =_\beta \overline{s'} \overline{s''}(\overline{s''}(t_2[v := t_1][v' := t_3]) \equiv \overline{s} \overline{s'} \overline{s''}(t_2[v := t_1][v' := t_3]) \equiv t$.

  - Moreover, $t \equiv \overline{s}(t_2[t_2[v := t_1][v' := t_3]) \equiv t' \equiv \overline{s'}(\overline{s''}(t_2[t_2[v := t_1][v' := t_3]) \equiv \overline{s'}(\overline{s''}(t_2[v := t_3]) \equiv t'$. Hence by IH, $t =_\beta t''$.

  - Now, $t'' \rightsquigarrow_\beta \overline{s_1} \overline{s'} \overline{s''}(t_2[v := t_3])$, but by BC, $v, v' \not\in \text{FV}(t_1)$ or $\text{FV}(t_3)$. Hence, by IH and substitution,

    $t'' =_\beta \overline{s_1} \overline{s'} \overline{s''}(t_2[v := t_3]) \equiv t'''$. Hence $t =_\beta t'$.

Therefore, $t =_\beta t''$, $t'' =_\beta t'''$, and $t' =_\beta t'''$, hence $t =_\beta t'$.

**Corollary 3.9** If $t \rightsquigarrow_\beta t'$ then $t =_\beta t'$.

**Theorem 3.10** (The general Church Rosser theorem)

If $t \rightsquigarrow_\beta t_1$ and $t \rightsquigarrow_\beta t_2$, then there exists $t_3$ such that $t_1 \rightsquigarrow_\beta t_3$ and $t_2 \rightsquigarrow_\beta t_3$.

**Proof:** As $t \rightsquigarrow_\beta t_1$ and $t \rightsquigarrow_\beta t_2$ then by Corollary 3.9, $t =_\beta t_1$ and $t =_\beta t_2$. Hence, $t_1 =_\beta t_2$ and by the Church Rosser property for the classical lambda calculus, there exists $t_3$ such that $t_1 \rightarrow_\beta t_3$ and $t_2 \rightarrow_\beta t_3$. But, $t' \rightarrow_\beta t''$ implies $t' \rightsquigarrow_\beta t''$. Hence the Church-Rosser theorem holds for the general $\beta$-reduction.

For the proof of Subject Reduction, we need the following “shuffle lemma”.

**Lemma 3.11** $\Gamma \vdash \overline{s_1}(t_1 \delta) \overline{s_2} t_2 : \rho \iff \Gamma \vdash \overline{s_1} \overline{s_2}(t_1 \delta) t_2 : \rho$ where $\overline{s_2}$ is well-balanced and the binding variables in $\overline{s_2}$ are not free in $t_1$.

**Proof:** By induction on weight(\overline{s_2}).

- case weight(\overline{s_2}) = 0 then nothing to prove.

- case weight(\overline{s_2}) = 2, say $\overline{s_2} \equiv (t_3 \delta)(\rho_1 \lambda v)$. We use induction on weight(\overline{s_1}).
Suppose $w(s_1) = 0$. 
\[ \Rightarrow \text{ suppose } \Gamma \vdash (t_1 \delta)(t_3 \delta)(\rho_1 \lambda_\nu)t_2 : \rho \]

Using the Generation lemma three times, we obtain:

\begin{align*}
\Gamma &\vdash (t_3 \delta)(\rho_1 \lambda_\nu)t_2 : \rho' \rightarrow \rho \\
\text{Hence } &\Gamma \vdash t_1 : \rho' \quad \text{(Context, \rightarrow\text{-elimination, (1), (3))} (4) \\
\Gamma &\vdash (\rho_1 \lambda_\nu)t_2 : \rho_1 \rightarrow (\rho' \rightarrow \rho) \\
\text{Now, } &\Gamma \vdash (\rho_1 \lambda_\nu)(t_1 \delta)t_2 : \rho_1 \rightarrow \rho \quad \text{(\rightarrow\text{-introduction, (4))} (5) \\
\text{And so, } &\Gamma \vdash (t_3 \delta)(\rho_1 \lambda_\nu)(t_1 \delta)t_2 : \rho \quad \text{(\rightarrow\text{-elimination, (5), (2))} (6) \\
\Leftrightarrow &\text{ Suppose } \Gamma \vdash (t_3 \delta)(\rho_1 \lambda_\nu)(t_1 \delta)t_2 : \rho \\
\text{Using the Generation lemma three times we obtain:} \end{align*}

\begin{align*}
\Gamma &\vdash (\rho_1 \lambda_\nu)(t_1 \delta)t_2 : \rho_1 \rightarrow \rho \\
\Gamma &\vdash t_3 : \rho_1 \quad (6) \\
\Gamma &\vdash (\rho_1 \lambda_\nu) \vdash (t_1 \delta)t_2 : \rho \\
\Gamma &\vdash (\rho_1 \lambda_\nu) \vdash t_2 : \rho' \rightarrow \rho \\
\text{Hence, } &\Gamma \vdash (\rho_1 \lambda_\nu)t_2 : \rho_1 \rightarrow (\rho' \rightarrow \rho) \quad \text{\rightarrow\text{-introduction, (7))} (9) \\
\Gamma &\vdash (t_3 \delta)(\rho_1 \lambda_\nu)t_2 : \rho' \rightarrow \rho \quad \text{\rightarrow\text{-elimination, (6), (9))} (10) \\
\Gamma &\vdash t_1 : \rho' \quad \text{(context, (8), as \nu \not\in FV(t_1))} (11) \\
\Gamma &\vdash (t_1 \delta)(t_3 \delta)(\rho_1 \lambda_\nu)t_2 : \rho \quad \text{\rightarrow\text{-elimination, (10), (11))} (12) \\
\end{align*}

Now suppose $w(s_1) = n + 1$. 

* Case $s_1 \equiv (t_3 \delta)s_1$ then 

\begin{align*}
\Gamma &\vdash (t_1 \delta)\overrightarrow{t}(t_3 \delta)(\rho_1 \lambda_\nu)t_2 : \rho \Leftrightarrow \text{Generation; \rightarrow\text{-elimination} \\
\Gamma &\vdash \overrightarrow{t}(t_1 \delta)(t_3 \delta)(\rho_1 \lambda_\nu)t_2 : \rho' \rightarrow \rho \land \Gamma \vdash t_4 : \rho' \Leftrightarrow \text{IH} \\
\Gamma &\vdash \overrightarrow{t}(t_3 \delta)(\rho_1 \lambda_\nu)(t_1 \delta)t_2 : \rho' \rightarrow \rho \land \Gamma \vdash t_4 : \rho' \Leftrightarrow \text{\rightarrow\text{-elimination}; Generation} \\
\Gamma &\vdash (t_1 \delta)\overrightarrow{t}(t_3 \delta)(\rho_1 \lambda_\nu)(t_1 \delta)t_2 : \rho \\
\end{align*}

* Case $s_1 \equiv (\rho_2 \lambda_\nu)s_1$ then 

\begin{align*}
\Gamma &\vdash (\rho_2 \lambda_\nu)\overrightarrow{s}(t_1 \delta)(t_3 \delta)(\rho_1 \lambda_\nu)t_2 : \rho \Leftrightarrow \text{Generation; \rightarrow\text{-introduction} \\
\Gamma &\vdash \overrightarrow{s}(t_1 \delta)(t_3 \delta)(\rho_1 \lambda_\nu)t_2 : \rho_3 \land \rho \equiv \rho_2 \rightarrow \rho_3 \Leftrightarrow \text{IH} \\
\Gamma &\vdash \overrightarrow{s}(t_3 \delta)(\rho_1 \lambda_\nu)(t_1 \delta)t_2 : \rho_3 \land \rho \equiv \rho_2 \rightarrow \rho_3 \Leftrightarrow \text{\rightarrow\text{-introduction}; Generation} \\
\Gamma &\vdash (\rho_2 \lambda_\nu)\overrightarrow{s}(t_3 \delta)(\rho_1 \lambda_\nu)(t_1 \delta)t_2 : \rho \\
\end{align*}

• case $w(s_2) = 2\left(\begin{array}{c}n+1 \end{array}\right), n \geq 1$. If $s_2 \equiv (t_3 \delta)s_3(s_1 \lambda_\nu)s_1$ where $s_3, s_4$ are well-balanced and IH holds for them, then: 

\begin{align*}
\Gamma &\vdash \overrightarrow{s}(t_1 \delta)(t_3 \delta)(\rho_1 \lambda_\nu)s_1t_2 : \rho \Leftrightarrow \text{IH} \\
\Gamma &\vdash \overrightarrow{s}(t_1 \delta)s_3(t_3 \delta)(\rho_1 \lambda_\nu)s_4t_2 : \rho \Leftrightarrow \text{IH} \\
\Gamma &\vdash \overrightarrow{s}(t_3 \delta)(\rho_1 \lambda_\nu)s_4t_2 : \rho \Leftrightarrow \text{IH} \\
\end{align*}
\[
\begin{align*}
\Gamma \vdash \overline{s_1}(t_3 \delta)(p_1 \lambda \nu)(t_1 \delta) \overline{s_1} t_2 : \rho \leftrightarrow^t H \\
\Gamma \vdash \overline{s_1}(t_3 \delta)(p_1 \lambda \nu)(t_1 \delta) \overline{s_1} t_2 : \rho \leftrightarrow^t H \\
\Gamma \vdash \overline{s_1}(t_3 \delta)(p_1 \lambda \nu) \overline{s_1}(t_1 \delta) t_2 : \rho.
\end{align*}
\]

**Remark 3.12** Note that in Lemma 3.11 above, we insisted on the condition that the binding variables in \(s_1\) are not free in \(t_1\) in order to avoid cases such as moving \((\nu \delta)\) in \((t_0 \delta)(\rho \lambda \nu)(\nu \delta)\) to the left of \((t_0 \delta)(\rho \lambda \nu)\).

Now we can prove Subject Reduction for generalised \(\beta\)-reduction.

**Theorem 3.13** (Generalised Subject Reduction)

If \(\Gamma \vdash t : \rho\) and \(t \rightarrow^\beta t'\) then \(\Gamma \vdash t' : \rho\).

**Proof:** By induction on \(\rightarrow^\beta\).

- **Basic Case:** \((t_1 \delta) \overline{s_2}(p_1 \lambda \nu) t_2 \rightarrow^\beta \overline{s_2}(t_2[v := t_1])\) and \(\Gamma \vdash (t_1 \delta) \overline{s_2}(p_1 \lambda \nu) t_2 : \rho \Rightarrow \text{Lemma 3.11}\)
- \(\Gamma \vdash (t_1 \delta) \overline{s_2}(p_1 \lambda \nu) t_2 : \rho \Rightarrow \text{Lemma 2.17}\)
- \(\Gamma \vdash \overline{s_2}(t_2[v := t_1]) : \rho\).

- The reflexivity, transitivity and compatibility cases are easy.

For Unicity of Types, we just need the following lemma:

**Lemma 3.14** If \(t \equiv^\beta t'\) then \(t =_\beta t'\).

**Proof:** By induction on \(t \equiv^\beta t'\) using Corollary 3.9.

**Lemma 3.15** (Generalised Unicity of Types)

1. \(\forall \Gamma \forall \nu \forall _{\rho', \rho} [\Gamma \vdash t : \rho \land \Gamma \vdash t : \rho' \Rightarrow \rho = \rho']\)
2. \(\forall \Gamma \forall t, t' \forall _{\rho, \rho'} [\Gamma \vdash t : \rho \land \Gamma \vdash t' : \rho' \land t \equiv^\beta t' \Rightarrow \rho = \rho']\)

**Proof:** The proof of 1 is the same for Lemma 2.18. The proof of 2 is also carried from Lemma 2.18 using Lemma 3.14 above.

Now we come to the proof of Strong Normalisation. For this, we need the following definition:

**Definition 3.16**

- We say that \(t \in \Lambda_T\) is strongly normalising with respect to \(\rightarrow^\beta\) iff every reduction path (with respect to \(\rightarrow^\beta\)) starting at \(t\), terminates.
- We define \(SN = \{t \in \Lambda_T : t\) is strongly normalising with respect to \(\rightarrow^\beta\}\).
- For \(A, B \subseteq \Lambda_T\) we define \(A \longrightarrow B = \{t \in \Lambda_T : \forall t' \in A[(t' \delta)t \in B]\}\).
- We define \(|\cdot| : \mathcal{T} \rightarrow \text{Power Set of } \Lambda_T\) as follows:

\[
\begin{align*}
|\nu| &= SN \\
|\rho \rightarrow \rho'| &= |\rho| \rightarrow |\rho'|
\end{align*}
\]

- We call \(X \subseteq SN\) saturated iff:
1. $\forall n \geq 0, t_1, \cdots t_n \in SN, v \in V[(t_1 \delta) \cdots (t_n \delta) v \in X]$.  
2. $\forall n \geq 0, t, t_1, \cdots, t_n \in SN, \rho \in T, \sigma$ well-balanced, $t' \in \Lambda_T$  
   $[(t_1 \delta) \cdots (t_n \delta) \sigma(t'[v := t]) \in X \Rightarrow (t_1 \delta) \cdots (t_n \delta) \sigma(\rho \lambda_\nu)(t') \in X]$.

- We define $SAT = \{X \subseteq \Lambda_T : X$ saturated$\}$

Those familiar with the proof of Strong Normalisation of $\lambda_\rightarrow$, will notice that we have accommodated $\Rightarrow_\beta$ in the definition of $SN$ and that in the second condition of a saturated set, we have accommodated extended redexes. The accommodation of saturated sets with extended redexes is not necessary, the proof can go without it. Furthermore, the following is the crucial $/3/. Easy$ induction on the generation of $/2/. A; B$ 

Lemma 3.17

1. $SN \in SAT$.
2. $A, B \in SAT \Rightarrow A \rightarrow B \in SAT$.
3. $\rho \in T \Rightarrow |\rho| \in SAT$.

Proof:

1. $SN \subseteq SN$ and if $t_1, \cdots, t_n \in SN, v \in V$ then similarly $(t_1 \delta) \cdots (t_n \delta) v \in SN$.  
   Now, if $t, t_1, \cdots, t_n \in SN, \rho \in T, \sigma$ is well-balanced and $t' \in \Lambda_T$ such that  
   $(t_1 \delta) \cdots (t_n \delta) \sigma(t'[v := t]) \in SN$ then also $(t_1 \delta) \cdots (t_n \delta) \sigma(\rho \lambda_\nu)(t') \in SN$:
   - Reductions inside $t', t, \sigma$ or one of the $t_i$ must terminate since these terms are $SN$  
     (subterms of $SN$-terms are themselves $SN$, $t'[v := t]$ is $SN \Rightarrow t'$ is $SN$).
   - A reduction path of $(t_1 \delta) \cdots (t_n \delta) \sigma(\rho \lambda_\nu) t'$ goes to $(t'_1 \delta) \cdots (t'_n \delta) \sigma(t''[v := t''])$ with $t' \Rightarrow_\beta t''$ etc. and then to $(t'\delta) \cdots (t''\delta) \sigma(t''[v := t''])$; since  
     $(t_1 \delta) \cdots (t_n \delta) \sigma(t'[v := t]) \in SN$ also $(t_1 \delta) \cdots (t_n \delta) \sigma(t''[v := t'']) \in SN$.

2. Suppose $A, B \in SAT$.
   - As $v \in A$ for all $v \in V$, we see: $t \in A \rightarrow B \Rightarrow (v \delta)t \in B \Rightarrow (v \delta)t \in SN \Rightarrow t \in SN$. So $A \rightarrow B \subseteq SN$.
   - If $t_1, \cdots, t_n \in SN, v \in V$ then for all $t \in A$, as $t \in SN$ and $B \in SAT$, we get that  
     $(t \delta)(t_1 \delta) \cdots (t_n \delta) v \in B$. Hence $(t_1 \delta) \cdots (t_n \delta) v \in A \rightarrow B$ which proves condition 1 of saturation.
   - As to condition 2, suppose $t, t_1, \cdots, t_n \in SN, t' \in \Lambda_T, \sigma$ is well-balanced, $\rho$ a term  
     and $(t_1 \delta) \cdots (t_n \delta) \sigma(t'[v := t]) \in A \rightarrow B$.
     Let $t'' \in A$. Then $(t'' \delta)(t_1 \delta) \cdots (t_n \delta) \sigma(t'[v := t]) \in B$, by definition of $A \rightarrow B$.  
     Hence $(t'' \delta)(t_1 \delta) \cdots (t_n \delta) \sigma(\rho \lambda_\nu) t' \in B$ since $B \in SAT, t'' \in A \subseteq SN$.  
     This means $(t_1 \delta) \cdots (t_n \delta) \sigma(\rho \lambda_\nu) t' \in A \rightarrow B$.

3. Easy induction on the generation of $\rho$ using 1 and 2.  

\[ \square \]
Corollary 3.18 For all $\rho \in \mathcal{T}$, we have $|\rho| \neq \emptyset$ and $|\rho| \subseteq \text{SN}$.

Proof: Note that no saturated set is empty (use $\text{SN} \neq \emptyset$ and condition 1 of saturated sets).

Definition 3.19

- A valuation is a map $g : V \rightarrow \Lambda_T$
- If $g$ is a valuation then $|g|_g$ is defined inductively as follows:
  
  $|v|_g = g(v)$
  
  $|(t \delta)t'|_g = (|t'_g|)g$
  
  $|((\rho \lambda v)t)|_g = (|\rho \lambda v|)g(v := v)$

where $g(v := N)$ is the valuation that assigns $g(v')$ to $v' \neq v$ and $N$ to $v$. Note that $|\rho|_g$ substitutes $g(v')$ for $v'$ in $t$ for all free variables $v'$ of $t$. For example, $|(\rho \lambda x)(g\delta)x|_g = (\rho \lambda x)(g(v)\delta)x$.

- $\models$ is defined as follows:
  
  $g \models t : \rho \iff |g|_g \in |\rho|$
  
  $g \models \Gamma \iff$ for all $(\rho \lambda v) \in \Gamma$, we have $g \models v : \rho$
  
  $\Gamma \models t : \rho \iff$ for all valuations $g$, if $g \models \Gamma$ then $g \models t : \rho$

Lemma 3.20 (Soundness)

If $\Gamma \models t : \rho$ then $\Gamma \models t : \rho$.

Proof: By a straightforward induction on the derivation of $\Gamma \vdash t : \rho$. We only treat the $\rightarrow$-introduction.

Suppose $\Gamma \vdash (\rho \lambda v)t : \rho \rightarrow \rho'$ out of $\Gamma(\rho \lambda v) \vdash t : \rho'$.

Suppose $g \models \Gamma$ in order to show $g \models (\rho \lambda v)t : \rho \rightarrow \rho'$ (i.e. for all $t' \in |\rho|$: $t'(\delta)(|\rho \lambda v|)g(v := v) \in |\rho'|$).

Let $t' \in |\Gamma|$. Then $g(v := t') \models \Gamma(\rho \lambda v)$, so by the induction hypothesis $|g(v := t')|_g \in |\rho'|$.

Since $t'(\delta)(|\rho \lambda v|)g(v := v) \models t'(\delta)(|\rho \lambda v|)g(v := v) \models t' \models v = t$ and $|\rho| \subseteq \text{SN}$ and $|\rho| \in \text{SAT}$, also $(t'(\delta)|)(\rho \lambda v)|_g \in |\rho'|$.

Theorem 3.21 (Strong Normalisation with respect to $\rightsquigarrow_{\beta}$)

If $\Gamma \vdash t : \rho$ then $t$ is strongly normalising with respect to $\rightsquigarrow_{\beta}$.

Proof: Suppose $\Gamma \vdash t : \rho$. Define $g(v) = v$. Then $g \models \Gamma$ (because $|\rho| \in \text{SAT}$, so $v \subseteq |\Lambda|$).

Hence by soundness $|\rho|_g \subseteq |\rho| \subseteq \text{SN}$. But $|\rho|_g \equiv t$.

\section{Term reshuffling}

In this section we shall rewrite terms so that all the newly visible redexes (obtained as a result of our item notation), can be subject to the ordinary classical $\beta$-reduction $\rightarrow_{\beta}$. We shall show that this term rewriting is correct and preserves both reduction (be it only in a certain sense) and typing.

Let us go back to the definition of $\delta \lambda$-couples. Recall that if $\Phi \equiv s_1 \cdots s_m$ for $m > 1$ where $s_1 s_m$ is a $\delta \lambda$-couple then $s_2 \cdots s_{m-1}$ is a well-balanced segment, $s_1 \equiv (t_1 \delta)$ is the $\delta$-item of the $\delta \lambda$-couple and $s_m \equiv (\rho \lambda v)$ is its $\lambda$-item. Now, we can move $s_1$ in $\Phi$ so that it occurs adjacent to $s_m$. That is, we may rewrite $\Phi$ as $s_2 \cdots s_{m-1} s_1 s_m$. 

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Note that we have assumed $\emptyset$ well-balanced. We assume it moreover non-bachelor.
Corollary 4.5 For each non-empty segment $\bar{s}$, there is a unique partitioning in segments $\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_n$, such that

1. $\bar{s} \equiv \bar{s}_0 \bar{s}_1 \cdots \bar{s}_n$,
2. For all $0 \leq i \leq n$, $\bar{s}_i$ is well-balanced in $\bar{s}$ for even $i$ and $\bar{s}$ is bachelor in $\bar{s}$ for odd $i$.
3. each bachelor $\lambda$-segment $\bar{s}_i$ precedes each bachelor $\delta$-segment $\bar{s}_k$ in $\bar{s}$.
4. $\bar{s}_{2n} \not\equiv \emptyset$ for $n > 0$. 
\[
\]

Example 4.6 $\bar{s} \equiv (\rho_1 \lambda_v)(\rho_2 \lambda_v)(\rho_3 \lambda_v)(\rho_4 \lambda_v)(\rho_5 \lambda_v)(\rho_6 \lambda_v)(t_5 \delta)$ has the following partitioning:

- well-balanced segment $\bar{s}_0 \equiv \emptyset$,
- bachelor segment $\bar{s}_1 \equiv (\rho_1 \lambda_v)(\rho_2 \lambda_v)$,
- well-balanced segment $\bar{s}_2 \equiv (t_1 \delta)(\rho_3 \lambda_v)$,
- bachelor segment $\bar{s}_3 \equiv (\rho_4 \lambda_v)(t_2 \delta)$,
- well-balanced segment $\bar{s}_4 \equiv (t_3 \delta)(t_4 \delta)(\rho_5 \lambda_v)(\rho_6 \lambda_v)$,
- bachelor segment $\bar{s}_5 \equiv (t_5 \delta)$.

4.2 The reshuffling procedure and its properties

In what follows, we use $\omega_1, \omega_2, \ldots$ to range over $\{\delta \} \cup \{\lambda_v; v \in V\}$, and we shall use $A_1, A_2, \ldots$ to range over both terms and types (i.e. over $A_T \cup T$).

Definition 4.7 $TS$ and $T$ are defined mutually recursively such that:

- $TS(\rho) =_{df} \rho$
- $TS(\bar{s}) =_{df} TS(\bar{s})$ if $\bar{s}$ is well-balanced
- $TS((A_1 \omega_1) \cdots (A_n \omega_n)) =_{df} (TS(A_1) \omega_1) \cdots (TS(A_n) \omega_n)$ if $(A_1 \omega_1) \cdots (A_n \omega_n)$ is bachelor
- $TS(\bar{s}_0 \cdots \bar{s}_n) =_{df} TS(\bar{s}_0) \cdots TS(\bar{s}_n)$ if $\bar{s}_0 \cdots \bar{s}_n$ is the unique partitioning of Corollary 4.5
- $T(\bar{s}(t \delta), (\rho \lambda_t) \bar{\sigma}) =_{df} (t \delta)(\rho \lambda_t)T(\bar{s}, \bar{\sigma})$
- $T(\bar{s}, (t \delta) \bar{\sigma}) =_{df} T(\bar{s}(TS(t \delta), \bar{\sigma})$
- $T(\emptyset, \emptyset) =_{df} \emptyset$

Note that in this definition, we use $\bar{s}$ bachelor to mean $\bar{s}$ bachelor in $\bar{s}$.

The following lemma will be needed in the proofs:

Lemma 4.8

1. If $\bar{s}$ is well-balanced, then $T(\bar{s}_1, \bar{s}_2) \equiv TS(\bar{s})T(\bar{s}_1, \bar{s}_2)$.
2. If $(t \delta)$ matches $(\rho \lambda_t)$ in $\bar{\sigma}$ then $TS(\bar{\sigma}) \equiv TS(\bar{s}(t \delta))(\rho \lambda_t) \bar{\sigma}$.

Note here that, from BC, no binding variables of $\bar{s}$ are free in $t$.\footnote{Note here that, from BC, no binding variables of $\bar{s}$ are free in $t$.}
3. If $\overline{v}$ contains no items which are partnered in $t$ then $TS(\overline{v}t) \equiv TS(\overline{v})TS(t)$.

4. If $\overline{v}$ is bachelor in $\overline{v}t$ or is well-balanced, then $TS(\overline{v}t) \equiv TS(\overline{v})TS(t)$.

**Proof:** 1: by induction on $\text{weight}(\overline{v})$. Case $\text{weight}(\overline{v}) = 0$ then obvious.

Case $\overline{v} \equiv (t\delta)\overline{v}(\rho_\lambda v)\overline{s}$ then

$$T(\overline{v}t, (t\delta)\overline{v}(\rho_\lambda v)\overline{s}) \equiv T(\overline{v}T(\overline{v}t, (t\delta)\overline{v}(\rho_\lambda v)\overline{s})) \equiv IH$$

$$TS(\overline{v}t)T(\overline{v}T(\overline{v}t, (t\delta)\overline{v}(\rho_\lambda v)\overline{s})) \equiv TS(\overline{v}T(\overline{v}t, (t\delta)\overline{v}(\rho_\lambda v)\overline{s})) \equiv IH$$

$$TS(\overline{v})T(\overline{v}T(\overline{v}t, (t\delta)\overline{v}(\rho_\lambda v)\overline{s})) \equiv TS(\overline{v}T(\overline{v}t, (t\delta)\overline{v}(\rho_\lambda v)\overline{s})) \equiv IH$$

2: using 1. 3: let $t \equiv \overline{s}_0 \cdots \overline{s}_m$ and $\overline{v} \equiv \overline{s}_0 \cdots \overline{s}_m$ be partitionings. Use cases on $\overline{s}_0$ being empty or not and on $\overline{s}_m$ being bachelor or well-balanced. 4: This is a corollary of 3 above. □

The following lemma shows that $TS(t)$ changes all $\delta\lambda$-couples of $t$ to $\delta\lambda$-segments.

**Lemma 4.9** For every subterm $t'$ of a term $t$, the following holds:

1. $TS(t')$ is well-defined.

2. If $\overline{v} \equiv (t'^\delta)\overline{v}(\rho_\lambda v)$ is a subsegment of $t'$ where $\overline{v}$ is well-balanced, then $TS(\overline{v}) \equiv TS(\overline{v})T(\overline{v}t'^\delta)(\rho_\lambda v)$.

3. If $\overline{v} \equiv (A_1\omega_1)\cdots(A_n\omega_n)$ is bachelor in $t'$, then $TS(\overline{v}) \equiv (TS(A_1)\omega_1)\cdots(TS(A_n)\omega_n)$ is bachelor in $TS(t')$.

4. If $\overline{v}$ is a subsegment of $t'$ which is well-balanced, then $TS(\overline{v})$ is well-balanced.

**Proof:** By induction on $t$.

- Case $t \equiv v$ then $t$ is the unique subterm of $t$ and all 1-4 hold.

- Assume $t \equiv (A\omega)t_2$ where IH holds for $A$ if $A \equiv t_1$ and for $t_2$. Let $t'$ be a subterm of $t$. If $t'$ is a subterm of $t_1$ (for $A \equiv t_1$) or $t_2$ then use IH. If $t' \equiv t_2$ then:

  - Case $(A\omega)$ is bachelor then $TS(t) \equiv$ Lemma 4.8 (1) $(TS(A)\omega)TS(t_2)$. Here all 1-4 hold by IH on $A$ and $t_2$.

  - Case $A \equiv t_1 \land (t_1\delta)$ matches $(\rho_\lambda v)$ in $t$. I.e. $t \equiv (t_1\delta)\overline{v}(\rho_\lambda v)t_3$ then $TS(t) \equiv$ Lemma 4.8 (1,3) $TS(\overline{v})T(\overline{v}t_1)\delta(\rho_\lambda v)TS(t_3)$. Now use IH to show 1-4. □

**Lemma 4.10** For all variables $v$ and terms $t, t'$ we have:

$TS(t) \equiv TS(TS(t))$ and $TS(t[v := t']) \equiv TS(TS(t)[v := TS(t)'])$.

**Proof:** By induction on $t$ we show that for all subterms $t''$ of $t$, $TS(t''[v := t']) \equiv TS(TS(t''))$ and $TS(t'[v := t']) \equiv TS(TS(t')[v := TS(t')])$. □

Note that if $t \rightarrow^\beta t'$ and if all the $\delta\lambda$-couples in $t$ are $\delta\lambda$-segments, then it is not necessary that all the $\delta\lambda$-couples of $t'$ are $\delta\lambda$-segments. In other words, we can have $TS(t_1) \rightarrow^\beta t_2$ where $t_2 \not\equiv TS(t_2)$. For example, $(x_1\delta)(x_2\delta)((\rho_\lambda x_1)(\delta)(\rho_\lambda x_4) \rightarrow^\beta \gamma_1 \rightarrow (x_1\delta)(x_2\delta)(\rho_\lambda x_1)(\delta)(\rho_\lambda x_4)\gamma_2$. Following this remark, we show that in a sense, term reshuffling preserves $\beta$-reduction.

**Lemma 4.11** If $t, t' \in \lambda_\rightarrow$ and $t \sim^\beta t'$ then $(\exists t'')(TS(t) \rightarrow^\beta t'') \land TS(t'') \equiv TS(t')$. In other words, the following diagram commutes:
Proof: By induction on the general $\sim_\beta$.

- Case $t \equiv \overline{s}(t_1)\overline{\pi}(\rho \lambda_0)t_3 \sim_\beta t' \equiv \overline{s}(t_2[v := t_1])$, we use induction on the number $n$ of bachelor $\delta$-items of $\overline{s}$ that are partnered in $t_3$. Recall that $\overline{s}$ is well-balanced.
  - Case $n = 0$ then
    \[
    TS(\overline{s}(t_1)\overline{\pi}(\rho \lambda_0)t_3) \quad \equiv \text{Lemma 4.8 (3.1)}
    \]
    \[
    TS(\overline{s}(t_1)\overline{\pi}(\rho \lambda_0))TS(t_3) \quad \equiv \text{Lemma 4.8 (2.1)}
    \]
    \[
    TS(\overline{s}(t_1)T(\overline{s}(t_1))\overline{\pi}(\rho \lambda_0))TS(t_3) \quad \to_\beta
    \]
    \[
    TS(\overline{s}(t_1)T(\overline{s}(t_1))\overline{\pi}(\rho \lambda_0)[v := TS(t_1)]) \quad \equiv t'.
    \]
    \[
    TS(t'') \quad \equiv \text{Lemma 4.9, 4.10, 4.8 (3.1)}
    \]
    \[
    TS(\overline{s}(t_1)\overline{\pi}(\rho \lambda_0)t_2)[v := TS(t_1)]) \quad \equiv \text{Lemma 4.10}
    \]
    \[
    TS(\overline{s}(t_1)\overline{\pi}(\rho \lambda_0)t_2[v := t_1])
    \]
    \[
    TS(\overline{s}(t_1)\overline{\pi}(\rho \lambda_0)t_2[v := t_1])
    \]
  - Assume the property holds for $n$ and let us show it for the case where $\overline{s}$ contains $n + 1$ $\delta$-items which match $\lambda$-items of $t_3$. Let $(t'' \delta)$ be the leftmost such $\delta$-item of $\overline{s}$. Take $\overline{s} \equiv \overline{s}(t'' \delta)\overline{s}_1$ and $t_2 \equiv \overline{s}_2(\rho \lambda_\nu)t_2$ where $(t'' \delta)$ matches $(\rho \lambda_\nu)$. By Lemma 4.4, $(t'' \delta)\overline{s}_1(t_1)\overline{\pi}(\rho \lambda_0)\overline{s}_2(\rho \lambda_\nu)$ is well-balanced. Moreover, no item of $\overline{s}_1$ has a partner in $(t'' \delta)\overline{s}_1(t_1)\overline{\pi}(\rho \lambda_0)t_3$. As $\overline{s}_1(t_1)\overline{\pi}(\rho \lambda_0)\overline{s}_2(t'' \delta)(\rho \lambda_\nu)t_2 \sim_\beta \overline{s}_1(\overline{s}_2(t'' \delta)(\rho \lambda_\nu)t_2[v := t_1])$, we find by IH, $t''$ such that
    \[
    TS(\overline{s}(t_1)\overline{\pi}(\rho \lambda_0)\overline{s}_2(t'' \delta)(\rho \lambda_\nu)t_2) \to_\beta t'' \wedge
    \]
    \[
    TS(t'') \equiv TS(\overline{s}_1(\overline{s}_2(t'' \delta)(\rho \lambda_\nu)t_2[v := t_1]))
    \]

Now, $TS(\overline{s}(t'' \delta))$ is the wanted term because:

- $TS(\overline{s}(t'' \delta)) \equiv \text{Lemma 4.8 (4)} \equiv \text{Lemma 4.8 (4)}$
- $TS(\overline{s}(t'' \delta)) \equiv \text{Lemma 4.8 (2)}$
- $TS(\overline{s}(t'' \delta)) \equiv \text{Lemma 4.8 (2)}$
- $TS(\overline{s}(t'' \delta)) \equiv \text{Lemma 4.8 (2)}$, $BC$
- $TS(\overline{s}(t'' \delta)) \equiv \text{Lemma 4.8 (1)}, s_1, s_2$ well-balanced
- $TS(\overline{s}(t'' \delta)) \equiv \text{Lemma 4.8 (1)}, s_1, s_2$ well-balanced

- The proof of compatibility is technical. The difficult case is: $t \equiv (t_1)\delta_2$ and $t_2 \sim_\beta t'_2$.
  - Distinguish the cases: $(t_1)\delta_2$ is bachelor or non-bachelor in $t$.

Corollary 4.12 If $t \to_\beta t'$ then there exist $t_0, t_1, \cdots t_n$ such that

\[
[(t \equiv t_0) \wedge (TS(t_0) \to_\beta t_1) \wedge (TS(t_1) \to_\beta t_2) \wedge \cdots \wedge (TS(t_{n-1}) \to_\beta t_n) \wedge (TS(t_n) \equiv TS(t'))]
\]

Proof: By induction on $\to_\beta$.

- Case $t \sim_\beta t'$ use Lemma 4.11.
Finally, we show that term reshuffling preserves typing:

Lemma 4.13 If $\Gamma \vdash t : \rho$ then $\Gamma \vdash TS(t) : \rho$.
Proof: By induction on $t$.

- Case $t \equiv v$, then nothing to prove.

- Case $t \equiv (\rho \lambda x_v)t_v$ then

\[
\begin{array}{ll}
\Gamma \vdash (\rho \lambda x_v)t' : \rho & \Rightarrow \text{Generation} \\
\Gamma (\rho \lambda x_v) \vdash t' : \rho \land \rho \equiv \rho \Rightarrow t'' \Rightarrow \text{IH} \\
\Gamma (\rho \lambda x_v) \vdash TS(t') : \rho \land \rho \equiv \rho \Rightarrow t'' \Rightarrow \rightarrow\text{-introduction, Lemma 4.8 (3)} \\
\Gamma \vdash TS((\rho \lambda x_v)t_v) : \rho
\end{array}
\]

- Case $t \equiv (t' \delta) t''$ then

  - Case $(t' \delta)$ is bachelor in $t$ then

\[
\begin{array}{ll}
\Gamma \vdash (t' \delta)t'' : \rho & \Rightarrow \text{Generation} \\
\Gamma \vdash t' : \rho \land \Gamma \vdash t'' : \rho & \Rightarrow \text{IH} \\
\Gamma \vdash TS(t') : \rho \land \Gamma \vdash TS(t'') : \rho & \Rightarrow \rightarrow\text{-elimination, Lemma 4.8 (3)} \\
\Gamma \vdash (TS(t') \delta)TS(t'') : \rho
\end{array}
\]

  - Case $(t' \delta)$ is partnered in $t$, then $t \equiv (t' \delta) \pi(\rho \lambda x_v)t_1$ where $\pi$ is well-balanced, and no binding variables of $\pi$ are free in $t'$.

\[
\begin{array}{ll}
\Gamma \vdash (t' \delta)\pi(\rho \lambda x_v)t_1 : \rho & \Rightarrow \text{Generation} \\
\Gamma \vdash t' : \rho \land \Gamma \vdash \pi(\rho \lambda x_v)t_1 : \rho & \Rightarrow \text{IH} \\
\Gamma \vdash TS(t') : \rho \land \Gamma \vdash TS(\pi(\rho \lambda x_v)t_1) : \rho & \Rightarrow \rightarrow\text{-elimination} \\
\Gamma \vdash (TS(t') \delta)TS(\pi(\rho \lambda x_v)t_1) : \rho & \Rightarrow \text{Lemma 4.8 (4)} \\
\Gamma \vdash (TS(t') \delta)TS(\pi(\rho \lambda x_v)TS(t_1)) : \rho & \Rightarrow \text{Lemma 3.11} \\
\Gamma \vdash TS(\pi(t') \delta)(\rho \lambda x_v)TS(t_1) : \rho & \Rightarrow \text{Lemma 4.8 (4)} \\
\Gamma \vdash TS(\pi(t') \delta)(\rho \lambda x_v)TS(t_1) : \rho & \Rightarrow \text{Lemma 4.8 (2)} \\
\Gamma \vdash TS((t' \delta)\pi(\rho \lambda x_v)t_1) : \rho & \Rightarrow \Gamma \vdash TS(t) : \rho
\end{array}
\]


5 Conclusion

In this paper, we observed that if we change slightly the classical $\lambda$-notation, then we can make more redexes visible. This is useful and is in line with current research on the needed redexes (for normal forms) as in [BKKS 87]. Making more redexes visible will work to our advantage if we could also contract these redexes before other ones. For example, in lazy evaluation ([Launchbury 93]), some redexes get frozen while other ones are being contracted. Now, if we had the ability of choosing which redex to contract out of all visible redexes, rather than waiting for some redex to be evaluated before we can proceed with the rest, then we can say that we have achieved a flexible system where we have control over what to contract rather than letting reductions force themselves in some order. This may lead to some advantages concerning optimal reductions as in [Lévy 80].

With our notation, and our new $\beta$-reduction, we achieve this flexibility and freedom of choice. Moreover, we do not lose any of the original properties. We have shown in fact that what we provide is a more general $\beta$-reduction where more redexes are visible and where all the original properties (using ordinary classical reduction) still hold for our general reduction. We believe this to be an important breakthrough which may lead to new reduction strategies that may explain various programming principles (such as lazy evaluation) in an elegant way.

We have shown further that, using item notation (which makes more redexes visible), one is able to stick to the old $\beta$-reduction and just do a simple reshuffling so that these newly visible redexes can be contracted before other redexes. We have shown that this reshuffling (which is very simple and can only be enabled in our notation), is correct. In fact, reshuffling does really make all redexes subject to immediate contraction and preserves typing. So, if $t$ has type $\rho$ then the reshuffled version of $t$ also has type $\rho$. It is moreover the case that if $t \leadsto_\beta t'$ using our extended reduction, then $TS(t)$ can be transformed into $TS(t')$ using classical reduction and intermediate term reshuffling.

The work carried out in this paper will have many applications. We mentioned the semantics of lazy evaluation and the new reduction strategies which may lead to further optimal results. These points are under investigation. The new notation moreover deserves attention. [KN 93] and [NK 94] have shown many of its advantages for formulating and generalising type theory and for rendering substitution explicit in the $\lambda$-calculus. Further advantages are also studied in [KN 9z].

References


